Sum-to-product identities

Sum-to-product identities let you convert in the opposite direction of product-to-sum identities. While product-to-sum identities let you take the product of trig functions and separate it into a sum or difference of trig functions, sum-to-product identities let you take the sum or difference of trig functions and combine them into a product.

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

These identities can be derived from the product-to-sum identities. We'll prove the first sum-to-product identity

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

by starting with the expression on the right side of the identity and deriving the expression on the left side from it. Using one of the product-to-sum identities, we have

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$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=2\left\{\left(\frac{1}{2}\right)\left[\sin\left(\frac{\theta+\alpha}{2}+\frac{\theta-\alpha}{2}\right)+\sin\left(\frac{\theta+\alpha}{2}-\frac{\theta-\alpha}{2}\right)\right]\right\}$$

Canceling the 2 and $1/2$ that are out in front, we simplify this to

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\left(\frac{\theta+\alpha}{2}+\frac{\theta-\alpha}{2}\right)+\sin\left(\frac{\theta+\alpha}{2}-\frac{\theta-\alpha}{2}\right)$$

Combining the terms within each set of parentheses gives

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\left[\frac{(\theta+\alpha)+(\theta-\alpha)}{2}\right]+\sin\left[\frac{(\theta+\alpha)-(\theta-\alpha)}{2}\right]$$

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\left[\frac{(\theta+\theta)+(\alpha-\alpha)}{2}\right]+\sin\left[\frac{(\theta-\theta)+(\alpha-(-\alpha))}{2}\right]$$

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\left[\frac{(2\theta+0)}{2}\right]+\sin\left[\frac{(0+2\alpha)}{2}\right]$$

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\left(\frac{2\theta}{2}\right)+\sin\left(\frac{2\alpha}{2}\right)$$

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=\sin\theta+\sin\alpha$$

And this is the first sum-to-product identity. Now we'll show some applications of these identities.

Example

Express $\cos(6\theta)-\cos(15\theta)$ as a product of two trig functions and a positive number a.

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By one of the sum-to-product identities,

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{6\theta + 15\theta}{2}\right)\sin\left(\frac{6\theta - 15\theta}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\sin\left(\frac{-9\theta}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\sin\left(-\frac{9\theta}{2}\right)$$

Using the odd identity for the sine function, we get

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\left[-\sin\left(\frac{9\theta}{2}\right)\right]$$

$$\cos(6\theta) - \cos(15\theta) = 2\sin\left(\frac{21\theta}{2}\right)\sin\left(\frac{9\theta}{2}\right)$$

This is in the prescribed form, with $a = 2$.

Let's do a different kind of example.

Example

Use one of the sum-to-product identities to prove the trig identity.

$$\sin\theta = \sin(\pi - \theta)$$

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We'll rewrite the given identity.

$$\sin\theta - \sin(\pi - \theta) = 0$$

$$\sin\theta = \sin(\pi - \theta)$$

By one of the sum-to-product identities,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta + \alpha}{2}\right)\sin\left(\frac{\theta - \alpha}{2}\right)$$

Setting α to $\pi - \theta$, we get

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left[\frac{\theta + (\pi - \theta)}{2}\right]\sin\left[\frac{\theta - (\pi - \theta)}{2}\right]$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left[\frac{(\theta + (-\theta)) + \pi}{2}\right]\sin\left[\frac{(\theta - (-\theta)) - \pi}{2}\right]$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{0 + \pi}{2}\right)\sin\left(\frac{2\theta - \pi}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{2\theta - \pi}{2}\right)$$

Since $\cos(\pi/2) = 0$, we find that

$$\sin\theta - \sin(\pi - \theta) = 2(0)\sin\left(\frac{2\theta - \pi}{2}\right) = 0$$

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Let's do another example.

Example

Use a sum-to-product identity to find the values of $\cos(14\pi/3) + \cos(13\pi/3)$.

Setting θ to $14\pi/3$ and α to $13\pi/3$ in the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

we have

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left[\frac{\left(\frac{14\pi}{3}+\frac{13\pi}{3}\right)}{2}\right]\cos\left[\frac{\left(\frac{14\pi}{3}-\frac{13\pi}{3}\right)}{2}\right]$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left[\frac{\left(\frac{27\pi}{3}\right)}{2}\right]\cos\left[\frac{\left(\frac{\pi}{3}\right)}{2}\right]$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{27\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

Reducing $27\pi/6$ to lowest terms:

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{9\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)$$

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We know that $9\pi/2$ is coterminal with $\pi/2$, so

$$\cos\left(\frac{9\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Therefore,

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2(0)\cos\left(\frac{\pi}{6}\right) = 0$$

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